# A state-dependent decomposition method for discrete-time open tandem queues

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Decomposition often is the only feasible and computationally efficient approach to compute steady-state performance measures for queueing networks. However, performance results may be subject to severe approximation errors since decomposition methods usually assume that the connecting stream can be approximated by renewal processes. To overcome the renewal assumption, we present the semi-Markov arrivals decomposition approach (SMAD). SMAD is a refined decomposition approach, where the connecting stream between the upstream and the downstream station is described by a semi-Markov process. Using this modelling approach, state-dependent inter-departure times from the upstream queue are preserved for downstream queuing analysis. Numerical results demonstrate that the approach produces accurate results, compared to simulation.

*Key words*: Decomposition; Point process; semi-Markov process

## 1. Introduction

We consider a discrete-time open tandem queue, where the upstream queue is of type $M/G/1$, and the downstream queue is of type $G/G/1$. Decomposition is often the only feasible and computationally efficient approach to compute steady-state performance measures in this type of queuing network. This approach partitions the network into individual queuing systems and analyses them in isolation. It is based on the assumption that the output stream of the upstream $M/G/1$-queue – which is fed into the downstream $GI/G/1$-queue – can be approximated by a renewal process. However, it is well known that the departure process is a point process that is generally difficult to deploy for queueing system analysis (Whitt 1981, 1982). Recently, it has been shown that performance results may be subject to severe approximation errors when applying the renewal decomposition method in this tandem queue (Jacobi and Furmans 2022). Thus, we outline a novel approach to overcome the renewal assumption. We use a semi-Markov arrival process to model the connecting stream between the upstream and the downstream queue.

## 2. Literature review

Decomposition approaches for open queueing networks generally rely on two basic assumptions (Govil and Fu 1999): First, it is assumed that the individual queueing systems in the network can be treated as being statistically independent. Second, it is assumed that the input to each queueing system is a renewal process. In the continuous-time domain, this approach was first by applied by Kuehn (1979) with modifications presented by Shanthikumar and Buzacott (1981), Whitt (1983), and Reiman (1990). In the discrete-time domain, Haßlinger and Rieger (1996) and Furmans (2004) proposed refinements of these so-called parametric decomposition methods which allows for the computation of the entire probability distributions of performance measures.

A crucial problem for decomposition methods in the continuous-time domain is the computation of the variability measures for the internal flows, and for the departure stream. The Queueing Network Analyzer (QNA) (Whitt 1983) employs two procedures (Whitt 1982) to approximate the point departure processes by renewal processes, the stationary interval method and the asymptotic method. Since neither of both methods yields promising results for a wide range of variability parameters, Whitt (1983) introduces a hybrid procedure based on the work by Albin (1984a,b). Finally, the variability of departure stream of the $GI/G/1$-queue is observed as an approximation of the stationary interval method (Whitt 1984).

Haßlinger and Rieger (1996) present a refinement of the parametric decomposition approach for the analysis of open queueing networks in the discrete-time domain. The discrete distribution of superpositions of renewal processes is reversibly obtained by the distribution of the minimum of the residual times of all superposed flows. A recursive method and a faster approach based on the z-transform for the computation of the stochastic split of a renewal process are presented. Despite discussing the renewal assumption and its implications, Haßlinger and Rieger (1996) state that "further study is needed to construct [...] representations of non-renewal processes" in the discrete-time domain that enable the computation of exact results.

## 3. Modelling approach

To overcome the renewal assumption for the analysis of tandem queues with Poisson arrivals and general service times, we introduce the semi-Markov decomposition approach (SMAD). The novelty of this decomposition method is that a semi-Markov process (SMP) is used to model the connecting stream between the upstream $M/G/1$- and the downstream $G/G/1$-queue. Let the stochastic process $\mathcal{Z} = \{(N_k, D_k), k = 1, 2, ...\}$ denote a semi-Markov process where $N \in \mathbb{N}_0$ is the number of customers in the upstream $M/G/1$-queue immediately after the departure instance of customer $k$, and $D_k \in \mathbb{N}$ is the inter-departure time between customers $k$ and $k + 1$. Let the probability function

$$f(t \,|\, i) = P(D = t \,|\, N = i) \tag{1}$$

denote the conditional probability that the inter-departure time is equal to $t$, given that the embedded Markov chain of the semi-Markov process $Z_k$ is in state $N_k = i$. The probability function $f(t \,|\, i)$ is equal to the service time, if the system is not empty immediately after the departure instance (that is, $i > 0$), and equal to the sum of the remaining inter-arrival time and the service time, if the system is starving after departure instance $k$ (that is, $i = 0$). For downstream queueing analysis, we deploy the discrete-time $SM/G/1$-queue, which has been introduced by Rieger and Haßlinger (1994).

## 4. Numerical results

We consider a tandem queue where the service time distributions are equal at the upstream and the downstream station, $P(B = 15) = P(B = 16) = 0.5$, and the arrival stream is defined by $\lambda = 0.0613$. The utilisation of the tandem queue is $\rho = 0.950$. We compute the probability distribution of waiting time at the downstream queue, and compare the results to the waiting time distributions obtained with the renewal decomposition approach and simulation. While the renewal decomposition method computes an expected waiting time $E(W) = 1.74$, SMAD computes an expected waiting time $E(W) = 2.11$, and simulation yields an expected waiting time $E(W) = 2.10$. We performed a Chi-Square Goodness-of-Fit Test and found a significant relationship between the waiting time distributions computed with SMAD, and simulation $\big(\chi^2(11; 612{,}682) = 5.69, p = .893\big)$.

## 5. Conclusion

Decomposition approaches for open queueing networks approximate the interconnecting streams as renewal processes. While this assumption allows for computationally efficient models, performance results obtained at downstream queues might be prone to considerable approximation errors. The novelty of SMAD is that a SMP is used to model the connecting stream between the upstream $M/G/1$- and the downstream $G/G/1$-queue. Thus, SMAD captures the state-dependent inter-departure times in the upstream $M/G/1$-queue departure process. While SMAD computes performance results with great accuracy, state space explosion of the embedded Markov chain in the SMP remains a concern. Thus, introducing a state space limit to increase the computationally efficiency is a natural extension to the method.

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